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SPECTRAL ANALYSIS OF CERTAIN COMPACT FACTORS FOR GAUSSIAN DYNAMICAL SYSTEMS

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ABSTRACT

For factors of a Gaussian automorphism T determined by compact subgroups of the group of unitary operators acting on L^2 of the spectral measure of T, we prove that the maximal spectral multiplicity is either 1 or infinity. As an application, we show that the maximal multiplicity of those factors in all $L^p, 1 < p < +\infty$, is the same.

Introduction

The theory of Gaussian automorphisms is one of, now classical, parts in ergodic theory (see [1]), being also a rich source or a tool of constructing interesting examples (e.g. [10], [11], [13], [14]). In contrast to that, very little seems to be known about the factors of Gaussian automorphisms. Only recently, J.-P.

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Thouvenot (during the conference in Podebrady 1994) showed that the Gaussian automophisms (and hence their factors) are disjoint from the class of automorphisms with minimal self-joinings.

In this paper, we make first steps in the study of factors of Gaussian automorphisms. We will deal with those determined by certain compact subgroups coming from the centralizer of the first chaos of $L^2(X,\mu)$. Our spectral analysis will allow us to extend the classical result—the multiplicity function is either 1 or is unbounded—to this family of factors (our proof is based on slightly modified ideas of the classical case). We would like to mention that for zero entropy Gaussian automorphisms (which is the only relevant case for the spectral analysis) there are no examples known of factor algebras which are not of the type that is described in our paper. Some measure-theoretic properties seem to resemble those of Gaussian automorphisms (embeddability in measurable flows, isomorphism of T and its inverse).

Given $T\colon (X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$ we consider the corresponding isometries U_T on each $L^p(X,\mu), 1 \leq p < +\infty$. Thouvenot asked how the (possibly infinite) number

$$K_p = \inf\{j \geq 1\colon (\exists f_1,\dots,f_j \in L^p(X,\mu))$$
$$\operatorname{span}\{f_kT^s\colon k=1,\dots,j, s\in\mathbb{Z}\} = L^p(X,\mu)\}$$

varies with p. Iwanik in [5], [6] showed that for each T with positive entropy and $p>1$, $K_p=+\infty$. In [7], it has been proved that for a Gaussian automorphism

$$K_{p_1} = K_{p_2} \quad \text{for each } p_1,p_2 > 1.$$

As an application of our spectral analysis of compact first chaos factors, we extend this result to these factors. The authors would like to thank A. Iwanik for helpful discussions.

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1. Preliminaries

Let $T\colon (X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$ be an ergodic automorphism of a standard Borel space. By a **factor** of T we mean a T-invariant σ-algebra $\mathcal{A}$ (more presisely, a factor of T is the quotient action of T on $(X/\sim,\mathcal{A},\mu)$, where for $x,y\in X$, $x\sim y$ if they cannot be distinguished by the sets of $\mathcal{A}$). By $C(T)$ we denote the

centralizer of T, i.e.

$$C(T) = \{S\colon (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)\colon ST = TS\}.$$

In this paper only S invertible will be considered. There is a natural topology (called the **weak topology**) on $C(T)$ given by

$$S_n \longrightarrow S \text{ if } (\forall A \in \mathcal{B}) \ \ \mu(S_n^{-1}A \triangle S^{-1}A) + \mu(S_n A \triangle SA) \longrightarrow 0.$$

T is said **to be embeddable in a measurable flow** if there exists an action $(T_t)_{t\in\mathbb{R}}$ on $(Y, \mathcal{E}, \nu)$ such that the map

$$\mathbb{R} \times X \ni (t, x) \mapsto T_t x \in X$$

is measurable and T is isomorphic to T_1. In this case (by identifying T and T_1), we find that $\{T_t\colon t \in \mathbb{R}\} \subset C(T)$. By WC$(T)$ we mean the closure (in the weak topology) of the set of all powers of T.

If $\mathcal{P} \subset C(T)$ then it determines a factor $\mathcal{A}(\mathcal{P})$ of T by the formula

$$\mathcal{A}(\mathcal{P}) = \{B \in \mathcal{B}\colon (\forall S \in \mathcal{P}) \ \ SB = B\}.$$

On the other hand, if $\mathcal{A} \subset \mathcal{B}$ is a factor then it induces a subgroup $\mathcal{P}(\mathcal{A})$ of $C(T)$,

$$\mathcal{P}(\mathcal{A}) = \{S \in C(T)\colon (\forall B \in \mathcal{A}) \ \ SB = B\}.$$

In the case when $\mathcal{P}$ is a compact subgroup of $C(T)$ we obtain (see [8], [15])

$$\mathcal{P}(\mathcal{A}(\mathcal{P})) = \mathcal{P} \tag{1}$$

and moreover T is a compact group extension of $\mathcal{A}(\mathcal{P})$.

Let $U_T\colon L^2(X,\mu) \longrightarrow L^2(X,\mu)$ be the associated unitary operator, $U_T f = fT$. Call σ_f the Borel measure on $\mathbb{T}$ given by

$$\hat{\sigma}_f(n) = \int_{\mathbb{T}} z^n \, d\sigma_f(z) = (U_T^n f, f)$$

the **spectral measure** of f. There exists $f \in L^2(X,\mu)$ such that σ_f dominates all other spectral measures; σ_T, the equivalence class of σ_f, is called the **maximal spectral type** of T. Up to spectral equivalence, U_T is described by σ_T and a function

$$M_T\colon \mathbb{T} \longrightarrow \{1, 2, \ldots, +\infty\},$$

defined σ_T a.e., called the **multiplicity function** of T; its essential supremum is said to be the **maximal spectral multiplicity** (see e.g. [12] for further details).

If $\mathcal{P} \subset C(T)$ is a compact subgroup then

$$L^2(\mathcal{A}(\mathcal{P})) = \{f \in L^2(X,\mu)\colon fS = f \ \ (\forall S \in \mathcal{P})\}. \tag{2}$$

In general, the maximal spectral type of T on $\mathcal{A}(\mathcal{P})$ is absolutely continuous with respect to σ_T and the multiplicity function is not bigger than M_T.

2. Factors given by compact subgroups of the centralizer of the first chaos

Put $X = \mathbb{R}^{\mathbb{Z}}$, $\mathcal{B}$ the σ-algebra of Borel sets and let $T\colon X \longrightarrow X$,

$$(Tx)_n = x_{n+1}, \quad n \in \mathbb{Z}.$$

Let $\pi(s)$ denote the projection onto the s-th coordinate.

A probability measure μ on $\mathcal{B}$ is said to be a **Gaussian** measure if

(i) $(\forall s \in \mathbb{Z})$ $\int_X \pi(s)\,d\mu = 0$,

(ii) $(\forall n, s_1, s_2 \in \mathbb{Z})$ $\int_X \pi(s_1)\pi(s_2)\,d\mu = \int_X \pi(s_1+n)\pi(s_2+n)\,d\mu$,

(iii) $(\forall s_1 < \ldots < s_r \in \mathbb{Z})$ $(\pi(s_1),\ldots,\pi(s_r))$ is an r-dimensional Gaussian vector, i.e.

$$\mu(\{x \in X\colon \pi(s_1)(x) \in C_1,\ldots,\pi(s_r)(x) \in C_r\}) = \int_{C_1\times\ldots\times C_r} p(t_1,\ldots,t_r)\,dt_1\ldots dt_r,$$

where

$$p(t_1,\ldots,t_r) = \text{const}\cdot e^{-\frac{1}{2}(D^{-1}t,t)}, \quad t = (t_1,\ldots,t_r)$$

and $D = [d_{ij}]$, with $d_{ij} = \int_X \pi(s_i)\pi(s_j)\,d\mu$.

Let σ be the spectral measure of the Gaussian automorphism $T\colon (X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$, i.e. a symmetric measure on the circle given by

$$\hat{\sigma}(n) = \int_{\mathbb{T}} z^n\,d\sigma(z) = \int_X \pi(n)\pi(0)\,d\mu = \int_X fT^nf\,d\mu, \quad n \in \mathbb{Z},$$

where $f(x) = \pi(0)(x)$. Throughout, σ is assumed to be continuous (equivalently, $T = T_\sigma$ is ergodic). We will use certain well-known spectral properties of

Gaussian automorphisms; the corresponding proofs can be found in [1]. There exists a decomposition of $L^2(X,\mu)$ into

$$L^2(X,\mu)=\bigoplus_{n=0}^{\infty}\mathcal{H}_c^{(n)},$$

where $\mathcal{H}_c^{(0)}$ denotes the subspace of constant functions and $\mathcal{H}_c^{(n)}=\mathcal{H}_r^{(n)}+i\mathcal{H}_r^{(n)}$ with $\mathcal{H}_r^{(n)}$ a real T-invariant space. We have

$$\mathcal{H}_r^{(1)}=\operatorname{span}\{\pi(s)\colon s\in\mathbb{Z}\}=\operatorname{span}\{\pi(0)T^s\colon s\in\mathbb{Z}\}.$$

The spaces $\mathcal{H}_r^{(n)}$, $n\geq 1$ are called the n-th **homogeneous chaos** of T. Moreover, $U_T\colon\mathcal{H}_c^{(n)}\longrightarrow\mathcal{H}_c^{(n)}$ is isomorphic to

$$V\colon L^2_{\mathrm{sym}}(\mathbb{T}^n,\sigma_n)\longrightarrow L^2_{\mathrm{sym}}(\mathbb{T}^n,\sigma_n),$$
$$VF(z_1,\dots,z_n)=z_1\cdot\ldots\cdot z_nF(z_1,\dots,z_n),$$

where $L^2_{\mathrm{sym}}(\mathbb{T}^n,\sigma_n)$ is the subspace of $L^2(\mathbb{T}^n,\sigma_n)$ of functions invariant under the coordinate action on $\mathbb{T}^n$ of the group $\mathcal{S}_n$ of permutations and $\sigma_n=\underbrace{\sigma\times\ldots\times\sigma}_{n}.$

We have

(3) the maximal spectral type of T on $\mathcal{H}_c^{(n)}$ is equal to $\sigma^{(n)}=\underbrace{\sigma*\ldots*\sigma}_{n}.$

PROPOSITION 1:

(i) *For each (real) unitary operator* $U\colon\mathcal{H}_r^{(1)}\longrightarrow\mathcal{H}_r^{(1)}$ *there exists a unique automorphism* $S\colon(X,\mathcal{B},\mu)\longrightarrow(X,\mathcal{B},\mu)$ *such that*

$$U_S|\mathcal{H}_r^{(1)}=U.$$

(ii) *If in addition* $UU_T=U_TU$ *then* $S\in C(T)$.

Proof: Let $Y_n=U\pi(n),n\in\mathbb{Z}$. The covariances (Y_n,Y_m) are equal to $(\pi(n),\pi(m))$ and (Y_n) is a Gaussian process (since all $Y_n\in\mathcal{H}_r^{(1)}$ and $\mathcal{H}_r^{(1)}$ consists solely of Gaussians elements), therefore the two processes are equal in law, i.e.

$$\begin{aligned}(4)\qquad &\mu(\{x\in X\colon(\pi(n_1)(x),\dots,\pi(n_k)(x))\in A\})\\ &=\mu(\{x\in X\colon(Y_{n_1}(x),\dots,Y_{n_k}(x))\in A\}).\end{aligned}$$

Moreover, the smallest σ-algebra generated by

$$\{Y_n^{-1}(C)\colon n\in\mathbb{Z},C\text{ a Borel subset of }\mathbb{R}\}$$

is equal to $\mathcal{B}$. If we now let

$$S(x) = (\dots, Y_0(x), Y_1(x), \dots)$$

then (4) says that S preserves μ and since $U_T(U\pi(i))(x) = U(\pi(i+1))(x)$, the result follows. ∎

Suppose that $S\colon (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$, $S \in C(T)$ and $U_S(\mathcal{H}_r^{(1)}) \subset \mathcal{H}_r^{(1)}$. Thus if $y_1, \dots, y_n \in \mathcal{H}_r^{(1)}$ then

$$:y_1 S \cdot \ldots \cdot y_n S :=: y_1 \cdot \ldots \cdot y_n : S, \tag{5}$$

where $: y_1 \dots y_n :$ denotes the corresponding Hermite–Itô polynomial of Gaussian variables $y_1, \dots, y_n$. Put

$$Q_r^{(m)} = \{F \in L^2(\mathbb{T}^m, \sigma_m)\colon F \text{ is invariant under } \mathcal{S}_m \text{ and } F(\overline{z_1}, \dots, \overline{z_n}) = \overline{F(z_1, \dots, z_n)}\}.$$

Let $VF(z_1, \dots, z_m) = z_1 \dots z_m F(z_1, \dots, z_m)$. We then have an isomorphism

$$\Theta_r^{(m)}\colon Q_r^{(m)} \longrightarrow \mathcal{H}_r^{(m)}$$

of V and T given by

$$\Theta_r^{(m)}\left(\sqrt{\frac{1}{m!}}\operatorname{Sym}(z_1^{s_1} \dots z_m^{s_m})\right) = :\prod_{k=1}^{m} \pi(s_k):, \quad s_1, \dots, s_m \in \mathbb{Z},$$

where $\operatorname{Sym}(F)(z_1, \dots, z_m) = \sum_{\tau \in \mathcal{S}_m} F(z_{\tau(1)}, \dots, z_{\tau(m)})$ ($\mathcal{H}_r^{(m)}$ is generated by the polynomials $: \prod_{k=1}^{m} \pi(s_k) :$).

Lemma 1: *Let $p(z) = \sum_{n=-N}^{N} a_n z^n$ ($a_n \in \mathbb{R}$). Then*

$$:\prod_{k=1}^{m} \pi(s_k)(p(T)) := \Theta_r^{(m)}\left(p(z_1) \dots p(z_m)\sqrt{\frac{1}{m!}}\operatorname{Sym}(z_1^{s_1} \dots z_m^{s_m})\right).$$

Proof: We have

$$:\prod_{k=1}^{m}\pi(s_k)(p(T)):=:\prod_{k=1}^{m}\left(\sum_{n=-N}^{N}a_n\pi(s_k+n)\right):$$

$$=:\sum\prod_{k=1}^{m}a_{n_k}\pi(s_k+n_k):=\sum\Theta_r^{(m)}\left(\sqrt{\frac{1}{m!}}\mathrm{Sym}(a_{n_1}z_1^{s_1+n_1}\dots a_{n_m}z_m^{s_m+n_m})\right)$$

$$=\Theta_r^{(m)}\left(\sqrt{\frac{1}{m!}}\mathrm{Sym}(\sum a_{n_1}z_1^{s_1+n_1}\dots a_{n_m}z_m^{s_m+n_m})\right)$$

$$=\Theta_r^{(m)}\left(\sqrt{\frac{1}{m!}}\mathrm{Sym}(z_1^{s_1}p(z_1)\dots z_m^{s_m}p(z_m))\right)$$

$$=\Theta_r^{(m)}\left(p(z_1)\dots p(z_m)\sqrt{\frac{1}{m!}}\mathrm{Sym}(z_1^{s_1}\dots z_m^{s_m})\right). \quad ∎$$

Since S acts on $\mathcal{H}_r^{(1)}$ and commutes with T, on its isomorphic copy $Q_r^{(1)}$ it will act by the formula

$$Sf(z)=g(z)f(z) \tag{6}$$

for a unique function $g\in L^2(\mathbb{T},\sigma)$ satisfying $|g(z)|=1$ σ-a.s. and $g(\overline{z})=\overline{g(z)}$.

PROPOSITION 2: *Suppose that $S\in C(T)$ and $U_S(\mathcal{H}_r^{(1)})\subset\mathcal{H}_r^{(1)}$. Then for each $m\geq 1$, $U_S(\mathcal{H}_r^{(m)})\subset\mathcal{H}_r^{(m)}$ and moreover (on $Q_r^{(m)}$) it acts by the formula*

$$SF(z_1,\dots,z_m)=g(z_1)\dots g(z_m)F(z_1,\dots,z_m).$$

Proof: Let $p_n=p_n(z)$ be a sequence of trigonometric polynomials such that $p_n\longrightarrow g$ in $Q_r^{(1)}$ (hence $p_n(T)\longrightarrow S$ strongly on $\mathcal{H}_r^{(1)}$). It is enough to show that in $Q_r^{(m)}$

$$S(\mathrm{Sym}(z_1^{s_1}\dots z_m^{s_m}))=g(z_1)\dots g(z_m)\mathrm{Sym}(z_1^{s_1}\dots z_m^{s_m}).$$

In view of (5) and Lemma 1, we have

$$:\pi(s_1)\dots\pi(s_m):S=\lim_{n\longrightarrow\infty}:\pi(s_1)(p_n(T))\dots\pi(s_m)(p_n(T)):$$

$$=\lim_{n\longrightarrow\infty}\Theta_r^{(m)}\left(p_n(z_1)\dots p_n(z_m)\sqrt{\frac{1}{m!}}\mathrm{Sym}(z_1^{s_1}\dots z_m^{s_m})\right)$$

$$=\Theta_r^{(m)}\left(g(z_1)\dots g(z_m)\sqrt{\frac{1}{m!}}\mathrm{Sym}(z_1^{s_1}\dots z_m^{s_m})\right)$$

and the result follows. ∎

REMARK 1: On $Q_c^{(m)} = Q_r^{(m)} + iQ_r^{(m)}$, S will act by the same formula. A sufficient condition for every $S \in C(T)$ to preserve $\mathcal{H}_r^{(1)}$ is, for example,

$$\sigma \perp \sum_{n=2}^{\infty} \frac{\sigma^{(n)}}{n!}$$

(indeed, if $f \in \mathcal{H}_c^{(1)}$ then fS has the same spectral measure, hence in view of (3), $fS \in \mathcal{H}_c^{(1)}$).

REMARK 2: If $S \in C(T)$ but S does not preserve $\mathcal{H}_r^{(1)}$, then the situation is more complicated. For example, we have the following result*:

$$(7) \qquad \begin{cases} \text{if } f \in \mathcal{H}_r = \bigoplus_{m=1}^{\infty} \mathcal{H}_r^{(m)} \text{ is a Gaussian variable} \\ \text{then either } f \in \mathcal{H}_r^{(1)} \text{ or } f = \sum_{i=1}^{\infty} f_i, f_i \in \mathcal{H}_r^{(i)} \\ \text{with infinitely many } f_i \text{ different from zero.} \end{cases}$$

Indeed, for all $n \geq 1$, for every $Y \in \bigoplus_{k=0}^{n} \mathcal{H}_c^{(k)} \setminus (\bigoplus_{k=0}^{n-1} \mathcal{H}_c^{(k)})$ there exists $\alpha_* \in \mathbb{R}_+$ such that

$$E[\exp(\alpha|Y|^{2/n})] < +\infty \quad \text{for } \alpha < \alpha_*,$$
$$E[\exp(\beta|Y|^{2/n})] = +\infty \quad \text{for } \beta > \alpha_*$$

(cf. for example [9]). Let f be a Gaussian variable with zero mean and variance $\sigma^2 > 0$; then for every $n > 1$ and every $\beta > 0$, $E[\exp(\beta|f|^{2/n})]$ is finite. By the above result, there is no $n > 1$ such that $f \in \bigoplus_{k=0}^{n} \mathcal{H}_c^{(k)}$.

We also have the following.

PROPOSITION 3: *Let S, S_n: $\mathcal{H}_r^{(1)} \longrightarrow \mathcal{H}_r^{(1)}$ be unitary operators, $S_n f(z) = g_n(z)f(z)$, $Sf(z) = g(z)f(z)$. Then the following statements are equivalent:*

1. $S_n \longrightarrow S$ *on* $\mathcal{H}_r^{(1)}$,
2. $g_n \longrightarrow g$ *in* $L^2(\mathbb{T}, \sigma)$,
3. $(\forall m \geq 1) \quad g_n(\cdot) \cdot \ldots \cdot g_n(\cdot) \longrightarrow g(\cdot) \cdot \ldots \cdot g(\cdot)$ *in* $L^2(\mathbb{T}^m, \sigma_m)$,
4. $(\forall m \geq 1) \quad S_n \longrightarrow S$ *on* $\mathcal{H}_r^{(m)}$,
5. $S_n \longrightarrow S$ *in* $C(T)$.

In particular, $\mathcal{P} \subset U(\mathcal{H}_r^{(1)})$ is compact as a subgroup of unitary operators iff the corresponding subgroup of $C(T)$ of unique extensions of all elements of $\mathcal{P}$ is compact in the weak topology.

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3. Spectral analysis of compact first chaos factors

Suppose that $\mathcal{P}$ is a compact subgroup of $C(T|\mathcal{H}_r^{(1)})$. Equivalently, $\mathcal{P}$ can be identified with a compact subgroup of functions $g \in L^2(\mathbb{T},\sigma)$ of modulus one (one considers pointwise multiplication as the group operation) satisfying additionally $g(\overline{z}) = \overline{g(z)}$, where the topology is given by $L^2(\mathbb{T},\sigma)$-convergence. If $\mathcal{A}(\mathcal{P})$ denotes the corresponding factor then

$$L^2(\mathcal{A}(\mathcal{P})) = \{f \in L^2(X,\mu)\colon fS = f \ \ (\forall S \in \mathcal{P})\} = \bigoplus_{m=0}^{\infty} \mathcal{H}_c^{(m)}(\mathcal{P})$$

and by our identification

$$\mathcal{H}_c^{(m)}(\mathcal{P}) \cong Q_c^{(m)}(\mathcal{P}) = \{F \in Q_c^{(m)}\colon (\forall g \in \mathcal{P})$$
$$g(z_1)\dots g(z_m)F(z_1,\dots,z_m) = F(z_1,\dots,z_m) \quad \sigma_m \text{ a.s.}\}.$$

Example 1: Assume that the constant function -1 belongs to $\mathcal{P}$. Then $L^2(\mathcal{A}(\mathcal{P})) \subset \bigoplus_{m=0}^{\infty} \mathcal{H}_c^{(2m)}$. We claim that if $\sigma^{(2n)} \perp \sigma^{(2m)}$ whenever $n \neq m$ then $\mathcal{A}(\mathcal{P})$ has no Gaussian factor. Indeed, suppose that a Gaussian automorphism T_η with spectral measure η appears as a factor of $\mathcal{A}(\mathcal{P})$. We have that

$$\eta + \frac{\eta^{(2)}}{2!} + \frac{\eta^{(3)}}{3!} + \dots \ll \sum_{n=1}^{\infty} \frac{\sigma^{(2n)}}{(2n)!}.$$

Given $n \geq 1$, let $\eta = \eta_n^{(1)} + \eta_n^{(2)}$ with $\eta_n^{(1)} \ll \sigma^{(2n)}$ and $\eta_n^{(2)} \perp \sigma^{(2n)}$. There must exist n such that $\eta_n^{(1)} \neq 0$. Hence, in $\mathcal{H}_r^{(1)}(T_\eta)$ there exists a Gaussian element whose spectral measure is equal to $\eta_n^{(1)}$. By isomorphism we have a one in $\mathcal{H}_r^{(2n)}(T_\sigma)$ which is a contradiction to (7).

Let $A_m(\mathcal{P}) = \{(z_1,\dots,z_m) \in \mathbb{T}^m\colon (\forall g \in \mathcal{P}) \ \ g(z_1)\dots g(z_m) = 1\}$. We have

(8) $$Q_c^{(m)}(\mathcal{P}) = \{F \in Q_c^{(m)}\colon F(z_1,\dots,z_m) = 0 \ \ \forall (z_1,\dots,z_m) \in (A_m(\mathcal{P}))^c\}.$$

Moreover,

(9) $$A_m(\mathcal{P}) \text{ is invariant under the action of } \mathcal{S}_m,$$

(10) $$A_{m_1}(\mathcal{P}) \times A_{m_2}(\mathcal{P}) \subset A_{m_1+m_2}(\mathcal{P}),$$

(11) $$(\exists m \geq 1) \quad \sigma_m(A_m(\mathcal{P})) > 0$$

(if not then $L^2(\mathcal{A}(\mathcal{P}))$ is trivial, hence T is a compact group extension of a one point dynamical system which is impossible since T is weakly mixing).

Put $p_m\colon \mathbb{T}^m \longrightarrow \mathbb{T}$, $p_m(z_1,\dots,z_m) = z_1\dots z_m$. We obviously have $\sigma_m p_m^{-1} = \sigma^{(m)}$ for $m \geq 1$. Disintegrate now σ_m over $\sigma^{(m)}$ to obtain

$$\sigma_m(\cdot) = \int_{\mathbb{T}} \nu_m(\cdot|\lambda)\, d\sigma^{(m)}(\lambda),$$

where $\nu_m(\cdot|\lambda)$ are the corresponding conditional measures concentrated on $p_m^{-1}(\lambda)$. Since $\sigma_m\{(z_1,\dots,z_m) \in \mathbb{T}^m\colon (\exists i,j) \quad z_i = z_j\} = 0$ (σ is continuous), for $\sigma^{(m)}$ a.e. $\lambda \in \mathbb{T}$ we have that $\nu_m(\cdot|\lambda)$ is concentrated on a number of points of $p_m^{-1}(\lambda)$ which is a multiple (possibly infinite) of $m!$. This observation will still be true if we consider the restriction of $\nu_m(\cdot|\lambda)$ to $A_m(\mathcal{P})$ in view of (9). For a Borel set $A \subset \mathbb{T}^m$ we put

$$\widetilde{p_m}(A) = \{\lambda \in \mathbb{T}\colon \nu_m(A|\lambda) > 0\}.$$

Notice that if $A \subset \mathbb{T}^m$ has positive σ_m-measure then $\sigma^{(m)}(\widetilde{p_m}(A))$ is also positive. What is more important: for each Borel set $A \subset \mathbb{T}^m$ we have

$$A \subset p_m^{-1}(\widetilde{p_m}(A)) \pmod{\sigma_m} \tag{12}$$

(indeed, we have

$$\sigma_m(A \setminus p_m^{-1}(\widetilde{p_m}(A)) = \int_{\mathbb{T}} \nu_m(A \setminus p_m^{-1}(\widetilde{p_m}(A))|\lambda)\, d\sigma^{(m)}(\lambda);$$

if $\lambda \notin \widetilde{p_m}(A)$, then even $\nu_m(A|\lambda) = 0$ while if $\lambda \in \widetilde{p_m}(A)$ then

$$\nu_m(A \setminus p_m^{-1}(\widetilde{p_m}(A))|\lambda) = \nu_m(A \cap p_m^{-1}(\lambda) \setminus p_m^{-1}(\widetilde{p_m}(A))|\lambda) = 0).$$

LEMMA 2: *Let $A \subset \mathbb{T}^m$ be Borel and invariant under S_m. Then the maximal spectral type of $VF(z_1,\dots,z_m) = z_1\dots z_m F(z_1,\dots,z_m)$ on*

$$\{F \in Q_c^{(m)}\colon F(z_1,\dots,z_m) = 0 \quad \text{for } (z_1,\dots,z_m) \in A^c\}$$

is equal to $\sigma^{(m)}|_{\widetilde{p_m}(A)}$.

Proof: Denote $f(\lambda) = \lambda$. Since $A \subset p_m^{-1}(\widetilde{p_m}(A))$, for $n \in \mathbb{Z}$,

$$\begin{aligned}(V^n F, F) &= \int_{\mathbb{T}^m} z_1^n \dots z_m^n |F(z_1, \dots, z_m)|^2 \, d\sigma_m(z_1, \dots, z_m) \\ &= \int_{p_m^{-1}(\widetilde{p_m}(A))} z_1^n \dots z_m^n |F(z_1, \dots, z_m)|^2 \, d\sigma_m(z_1, \dots, z_m) \\ &= \int_{\mathbb{T}^m} ((\chi_{\widetilde{p_m}(A)} \cdot f^n) \circ p_m(z_1, \dots, z_m) |F(z_1, \dots, z_m)|^2 \, d\sigma_m(z_1, \dots, z_m) \\ &= \int_{\mathbb{T}} \chi_{\widetilde{p_m}(A)}(\lambda) f(\lambda)^n E(|F|^2 | p_m)(\lambda) \, d\sigma^{(m)}(\lambda) \\ &= \int_{\mathbb{T}} \lambda^n E(|F|^2 | p_m)(\lambda) \, d\sigma^{(m)}|_{\widetilde{p_m}(A)}(\lambda).\end{aligned}$$

Finally,

$$E(\chi_A | p_m)(\lambda) = \int_{\mathbb{T}^m} \chi_A(z_1, \dots, z_m) \, d\nu_m(z_1, \dots, z_m | \lambda) = \nu_m(A|\lambda)$$

and the result follows. ∎

From this lemma it is clear that if there exists a set of positive $\sigma^{(m)}$ measure $B \subset \mathbb{T}, \sigma^{(m)}(B) > 0$ such that

$$\nu_m(\cdot|\lambda)|_{A_m(\mathcal{P})} \quad \text{is not discrete}$$

for each $\lambda \in B$, then ∞ is an essential value of the multiplicity function of V on $Q_c^{(m)}(\mathcal{P})$. Indeed, we can make a measurable choice from the fibers $p_m^{-1}(\lambda), \lambda \in B$ so that we obtain

$$A_1, A_2, \dots \subset A_m(\mathcal{P}), \quad A_i \cap A_j = \emptyset, \quad \nu_m(A_i|\lambda) > 0, \quad i, j = 1, 2, \dots, \ \lambda \in B$$

and moreover $\widetilde{p_m}(A_1) = \widetilde{p_m}(A_2) = \dots = B$.

COROLLARY 1: *The maximal spectral type of $\mathcal{A}(\mathcal{P})$ is equal to* $\sum_{m=0}^{\infty} \frac{\sigma^{(m)}|_{\widetilde{p_m}(A_m(\mathcal{P}))}}{m!}$. ∎

From now on, we assume that the conditional measures are discrete. Set

$$A_p^{(m)} = \{\lambda \in \mathbb{T}: \nu_m(\cdot|\lambda) \text{ is concentrated on } pm! \text{ points}\},$$

$p = 1, 2, \dots, +\infty$ and let $A_{m,p}(\mathcal{P}) = p_m^{-1}(A_p^{(m)}) \cap A_m(\mathcal{P})$. We have a natural partition

$$A_{m,p}(\mathcal{P}) = A_{m,p,1}(\mathcal{P}) \cup \dots \cup A_{m,p,p}(\mathcal{P}),$$

where

$$A_{m,p,i}(\mathcal{P}) = \{(z_1,\dots,z_m) \in p_m^{-1}(A_p^{(m)}) \cap A_m(\mathcal{P}): \nu_m(\cdot|p_m(z_1,\dots,z_m))|_{A_m(\mathcal{P})} \text{ is concentrated on } im! \text{ points}\}.$$

THEOREM 1: *The set of essential values of the multiplicity function M of V on $A_p^{(m)}(\mathcal{P})$ is equal to $\{1 \le i \le p: \sigma_m(A_{m,p,i}(\mathcal{P})) > 0\}$.*

Proof: Let $A_{m,p,i_1}(\mathcal{P}),\dots,A_{m,p,i_s}(\mathcal{P})$, $i_1 < \cdots < i_s$ be all sets $A_{m,p,i}(\mathcal{P})$ with positive σ_m measure. There exist disjoint sets $B_1,\dots,B_u$, $u = i_1+\cdots+i_s$ of positive measure such that $A_{m,p}(\mathcal{P}) = \bigcup_{j=1}^u B_j$ selected in the following way: each one of the sets $B_1,\dots,B_{i_1}$ chooses $m!$ atoms that differ one from another only by a permutation on coordinates from each fiber of $A_{m,p,i_1}(\mathcal{P}),\dots,A_{m,p,i_s}(\mathcal{P})$ and we have

$$\widetilde{p_m}(B_1) = \cdots = \widetilde{p_m}(B_{i_1}) = \widetilde{p_m}(A_{m,p}(\mathcal{P})).$$

Then each $B_{i_1+1},\dots,B_{i_1+i_2}$ choses $m!$ atoms as above from each fiber over

$$A_{m,p,i_2}(\mathcal{P}),\dots,A_{m,p,i_s}(\mathcal{P})$$

and we have

$$\widetilde{p_m}(B_{i_1+1}) = \cdots = \widetilde{p_m}(B_{i_1+i_2}) = \widetilde{p_m}(A_{m,p,i_2}(\mathcal{P}) \cup \cdots \cup A_{m,p,i_s}(\mathcal{P})),$$

etc. Finally each $B_{i_1+\cdots+i_{s-1}+1},\dots,B_{i_1+\cdots+i_s}$ choses $m!$ atoms as above from each fiber over $A_{m,p,i_s}(\mathcal{P})$ and we have $\widetilde{p_m}(B_{i_1+\cdots+i_{s-1}+1}) = \cdots = \widetilde{p_m}(B_{i_1+\cdots+i_s}) = \widetilde{p_m}(A_{m,p,i_s}(\mathcal{P}))$. Now, for each $1 \le r \le s$, the action of V on each of the invariant subspaces

$$\{F \in Q_c^{(m)}: F(z_1,\dots,z_m) = 0 \text{ for } (z_1,\dots,z_m) \in (B_{i_1+\dots+i_{r-1}+k})^c\},$$

$k = 1,\dots,i_r$, is isomorphic to the multiplication by λ on

$$\{f \in Q_c^{(1)}: f(\lambda) = 0 \text{ for } \lambda \in \widetilde{p_m}(A_{m,p,i_r}(\mathcal{P}) \cup \cdots \cup A_{m,p,i_s}(\mathcal{P}))^c\}$$

(with $\sigma^{(m)}$ restricted) and the result follows. ∎

Before we prove our main result, some auxiliary lemmas will be needed.

LEMMA 3: *If μ,ν are two finite measures on the circle then for any two Borel sets $A, B \subset \mathbb{T}$*

$$\nu|_A * \mu|_B \ll (\nu * \mu)|_{\widetilde{p_2}(A\times B)},$$

where for $C \subset \mathbb{T} \times \mathbb{T}$, $\widetilde{p_2}(C) = \{\lambda \in \mathbb{T}: \rho(C|\lambda) > 0\}$ *and*

$$(\nu \times \mu)(\cdot) = \int_{\mathbb{T}} \rho(\cdot|\lambda)\, d(\nu * \mu)(\lambda).$$

Proof: First notice that similarly as in the case of (12) we get that for $\nu * \mu$ a.a. $\lambda \in \mathbb{T}$

$$A \times B \subset p_2^{-1}(\widetilde{p_2}(A \times B)) \text{ with respect to } \rho(\cdot|\lambda). \tag{13}$$

Let $C \subset \mathbb{T}$ be a Borel set such that $(\nu * \mu)|_{\widetilde{p_2}(A\times B)}(C) = 0$. Hence

$$(\nu \times \mu)(p_2^{-1}(\widetilde{p_2}(A \times B) \cap C)) = 0,$$

so for $\nu * \mu$ a.a. $\lambda \in \mathbb{T}$, $\rho(p_2^{-1}(\widetilde{p_2}(A \times B) \cap C)|\lambda) = 0$. In view of (13), for $\nu * \mu$ a.a. $\lambda \in \mathbb{T}$, $\rho(A \times B \cap p_2^{-1}(C)|\lambda) = 0$ and consequently

$$\begin{aligned} 0 &= \int_{\mathbb{T}} \rho(A \times B \cap p_2^{-1}(C)|\lambda) d(\nu * \mu)(\lambda) = (\nu \times \mu)(A \times B \cap p_2^{-1}(C)) \\ &= (\nu|_A \times \mu|_B)(p_2^{-1}(C)) = (\nu|_A * \mu|_B)(C). \end{aligned}$$ ∎

Lemma 4: *Let* $(X, \mu) \xrightarrow{f} (Y, \nu) \xrightarrow{g} (Z, \gamma)$ *be standard Borel probability spaces with the corresponding disintegrations*

$$\mu = \int_Y \mu_y\, d\nu(y), \quad \nu = \int_Z \nu_z\, d\gamma(z), \quad \mu = \int_Z \tilde{\mu}_z\, d\gamma(z).$$

(i) *We have* $\tilde{\mu}_z(\cdot) = \int \mu_y(\cdot)\, d\nu_z(y)$ *for a.a.* $z \in Z$.

(ii) *If* $\mathcal{S}$ *is a finite group acting on* (X, μ) *and*

$$(\forall \tau, \tau' \in \mathcal{S}) \quad gf(\tau x) = gf(\tau' x),$$

then whenever x *is an atom of* $\tilde{\mu}_z$, *so are all* $\tau x, \tau \in \mathcal{S}$ $(x \in X)$.

Proof: For each $A \subset X$ we have

$$\mu(A) = \int_Y E(A|f)(y)\, d\nu(y) = \int_Z \left(\int_Y \mu_y(A)\, d\nu_z(y) \right) d\gamma(z)$$

and the uniqueness of disintegration yields the result; (ii) follows similarly. ∎

Lemma 5: *Under the above assumptions, suppose that there exists a set* $B \subset Y, \nu(B) > 0$ *such that the corresponding conditional measures* μ_y, $y \in B$ *on* $f^{-1}(y)$ *are purely atomic with* k *atoms. Assume that the conditional measures* $\tilde{\mu}_z$, $z \in Z$ *are also discrete. Then there exists a set* $C \subset Z, \gamma(C) > 0$ *such that* $B \subset g^{-1}(C)$ *and, if* $z \in C$, *then* $\tilde{\mu}_z$ *on* $(g \circ f)^{-1}(z)$ *has at least* k *atoms.*

Proof: Suppose that $x \in X$, $z \in Z$. From Lemma 4 it follows that

$$\tilde{\mu}_z(\{x\}) = \int_{g^{-1}(z)} \mu_y(\{x\})\, d\nu_z(y).$$

But x belongs to a unique fiber $f^{-1}(y_0)$, so

$$\tilde{\mu}_z(\{x\}) = \mu_{y_0}(\{x\})\nu_z(\{y_0\})$$

and, directly from this,

$$\tilde{\mu}_z(\{x\}) > 0 \quad \text{iff } \mu_{y_0}(\{x\}) > 0 \quad \text{and} \quad \nu_z(\{y_0\}) > 0.$$

Consequently, if x is an atom of $\tilde{\mu}_z$ then *all* atoms of μ_{y_0} will be atoms of $\tilde{\mu}_z$.

It is clear that $\mu(f^{-1}(B)) > 0$, whence there exists a set $C \subset Z$, $\gamma(C) > 0$ such that $\tilde{\mu}_z(f^{-1}(B)) > 0$ for each $z \in C$. For such z, $\tilde{\mu}_z$ is purely atomic, so there exists $x \in f^{-1}(B)$ such that $\tilde{\mu}_z(\{x\}) > 0$. Now, all atoms of μ_{y_0}, where $x \in f^{-1}(y_0)$, must be atoms of $\tilde{\mu}_z$. ∎

THEOREM 2: *If the spectrum of T on $\mathcal{A}(\mathcal{P})$ is not simple then its maximal spectral multiplicity on $L^2(\mathcal{A}(\mathcal{P}))$ is equal to infinity.*

Proof: We will first consider the case where the conditional measures $\nu_m(\cdot|\lambda)|_{A_m(\mathcal{P})}$ are discrete (for all m) and for some $m \geq 1$ on $Q_c^{(m)}(\mathcal{P})$, $\nu_m(\cdot|\lambda)|_{A_m(\mathcal{P})}$ is concentrated on $qm!$ points (with $q \geq 2$). We will show that then, on $Q_c^{(2m)}(\mathcal{P})$, a number t with $t \geq q^2$ is an essential value of the multiplicity function.

Consider

$$(\mathbb{T}^{2m}, \sigma_{2m}) \xrightarrow{p_m \times p_m} (\mathbb{T}\times\mathbb{T}, \sigma^{(m)} \times \sigma^{(m)}) \xrightarrow{p_2} (\mathbb{T}, \underbrace{\sigma^{(m)} * \sigma^{(m)}}_{\sigma^{(2m)}}),$$

where $p_m \times p_m(z_1, \ldots, z_m, z_{m+1}, \ldots, z_{2m}) = (z_1 \ldots z_m, z_{m+1} \ldots z_{2m})$. First notice that for $B_1, B_2 \subset \mathbb{T}^m$,

$$\begin{aligned}\sigma_{2m}(B_1 \times B_2) &= \sigma_m(B_1)\sigma_m(B_2)\\ &= \int_{\mathbb{T}} \nu_m(B_1|\lambda_1)\, d\sigma^{(m)}(\lambda_1) \int_{\mathbb{T}} \nu_m(B_2|\lambda_2)\, d\sigma^{(m)}(\lambda_2)\\ &= \int_{\mathbb{T}\times\mathbb{T}} \nu_m(B_1|\lambda_1)\nu_m(B_2|\lambda_2)\, d(\sigma^{(m)} \times \sigma^{(m)})(\lambda_1, \lambda_2),\end{aligned}$$

whence the corresponding conditional measures $\overline{\nu}_m(\cdot|(\lambda_1,\lambda_2))$ of the disintegration of σ_{2m} with respect to $\sigma^{(m)}\times\sigma^{(m)}$ are equal to $\nu_m(\cdot|\lambda_1)\times\nu_m(\cdot|\lambda_2)$.

Consider the sets

$$A_q^{(m)}\ni\lambda\mapsto F(\lambda)=\{(z_1,\ldots,z_m)\in\mathbb{T}^m\colon(\exists 1\le i\le m)(\exists(z_1',\ldots,z_m')\in\mathbb{T}^m)\\ z_i=z_i' \text{ and } (z_1',\ldots,z_m') \text{ is an atom of } \nu_m(\cdot|\lambda)|_{A_m(\mathcal{P})}\}.$$

We have $\sigma_m(F(\lambda))=0$. Consequently, given $\delta>0$ we can find a set $F^{(\delta)}\subset\mathbb{T}^m$ which is a measurable union of the sets $F(\lambda)$ and for which $0<\sigma_m(F^{(\delta)})<\delta$. (Indeed, let $\mathcal{R}_n=\{R_n^{(1)},\ldots,R_n^{(n)}\}$ be an increasing sequence of finite partitions tending to the point partition, set

$$F(R_n^{(i)})=\bigcup_{\lambda\in R_n^{(i)}}F(\lambda);$$

we have $\{\lambda\}=\bigcap_{n=1}^\infty R_n^{(i_{n_\lambda})}$, where $\lambda\in R_n^{(i_{n_\lambda})}$; moreover the map

$$\lambda\mapsto\nu_m(R_n^{(i)}|\lambda)$$

is measurable and hence we can find a λ such that this map (depending on n and i) is continuous at λ for all $n\ge1$ and $1\le i\le n$; for such a λ, given $\varepsilon>0$, all atoms of $\nu_m(\cdot|\lambda')|_{A_m(\mathcal{P})}$, where $|\lambda-\lambda'|<\eta$ must be within ε with the atoms of $\nu_m(\cdot|\lambda)|_{A_m(\mathcal{P})}$; since $(R_n^{(i_{n_\lambda})})_{n\ge1}$ forms a decreasing sequence of sets, $F(\lambda)=\bigcap_{n=1}^\infty F(R_n^{(i_{n_\lambda})})$ and $\sigma_m(F(\lambda))=0$, our claim follows.)

Given a natural number $k\ge1$ define

$$G_k=\{\lambda\in A_q^{(m)}\colon \text{minimal measure of the atoms of } \nu_m(\cdot|\lambda)|_{A_m(\mathcal{P})} \text{ is at least } \frac{1}{k}\}.$$

We have that the sequence (G_k) is an increasing sequence of measurable sets and $\bigcup_{k=1}^\infty G_k=A_q^{(m)}\cap\widetilde{p_m}(A_m(\mathcal{P}))$. Hence, there exists k_0 such that

$$\sigma_m(p_m^{-1}(G_{k_0}))\ge\frac{1}{2}\sigma_m(p_m^{-1}(A_q^{(m)}\cap\widetilde{p_m}(A_m(\mathcal{P})))).$$

If $B\subset\mathbb{T}^m$ is measurable,

$$J_{k_0}=\bigcup_{\lambda\in G_{k_0}}\operatorname{supp}(\nu_m(\cdot|\lambda))$$

(the support is meant in $A_m(\mathcal{P})$) and

$$\sigma^{(m)}(\{\lambda\in\widetilde{p_m}(A_m(\mathcal{P}))\colon\operatorname{supp}(\nu_m(\cdot|\lambda))\cap B\ne\emptyset\})\ge\varepsilon\sigma^{(m)}(A_q^{(m)}\cap\widetilde{p_m}(A_m(\mathcal{P}))),$$

then

$$\sigma_m(B\cap J_{k_0}) = \int_{\mathbb{T}} \nu_m(B\cap J_{k_0}|\lambda)\,d\sigma^{(m)}(\lambda) \geq \frac{\varepsilon}{k_0}\sigma^{(m)}(A_q^{(m)}\cap \widetilde{p_m}(A_m(\mathcal{P})))$$
$$= \frac{\varepsilon}{k_0}\sigma_m(p_m^{-1}(A_q^{(m)}\cap \widetilde{p_m}(A_m(\mathcal{P})))).$$

Therefore, if we have $\delta = \delta(\varepsilon, k_0)$ small enough, then we can find a set $\tilde{C}_1$ of positive σ_m-measure consisting of a union of $\operatorname{supp}\nu_m(\cdot|\lambda)$ for λ belonging to a subset of G_{k_0} such that if we let $\tilde{C}_2 = \bigcup_{\lambda\in S}\operatorname{supp}(\cdot|\lambda), F^{(\delta)} = F(S)$ then $\tilde{C}_1$ and $\tilde{C}_2$ (and their $\tilde{p_m}$-images) are disjoint. If $(z_1^{(i)},\dots,z_m^{(i)})\in\tilde{C}_i,\ i=1,2$ then this point must be an atom of $\nu_m(\cdot|\lambda_i)$ $(\lambda_i = p_m(z_1^{(i)},\dots,z_m^{(i)}))$ and consequently

$$(z_1^{(1)},\dots,z_m^{(1)},z_1^{(2)},\dots,z_m^{(2)})$$

is an atom of $\nu_m(\cdot|\lambda_1)\times\nu_m(\cdot|\lambda_2)$. Now, by the definition of $\tilde{C}_i,\ i=1,2$, Lemma 4(ii), Lemma 5 and (10), it follows that all permutations of the $2m$ coordinates give us pairwise different new atoms; (and since the number of obvious atoms is $(qm!)^2$) we conclude that the number of atoms is at least $(qm!)^2\cdot\binom{2m}{m} = q^2(2m)!$.

In order to complete the proof of Theorem 2 it is enough to show that

$$\text{(14)}\qquad \left\{\begin{array}{l}\text{if the maximal spectral types } \sigma^{(m)}|_{\widetilde{p_m}(A_m(\mathcal{P}))}\\ \text{are not disjoint then the multiplicity}\\ \text{function } M = M(\mathcal{A}(\mathcal{P})) \text{ is not bounded.}\end{array}\right.$$

We will first show that for each $m_1, m_2\in\mathbb{N}$

$$\text{(15)}\quad \sigma^{(m_1)}|_{\widetilde{p_{m_1}}(A_{m_1}(\mathcal{P}))} * \sigma^{(m_2)}|_{\widetilde{p_{m_2}}(A_{m_2}(\mathcal{P}))} \ll \sigma^{(m_1+m_2)}|_{\widetilde{p_{m_1+m_2}}(A_{m_1+m_2}(\mathcal{P}))}.$$

To this end consider

$$(\mathbb{T}^{m_1+m_2},\sigma_{m_1+m_2}) \overset{p_{m_1}\times p_{m_2}}{\longrightarrow} (\mathbb{T}\times\mathbb{T},\sigma^{(m_1)}\times\sigma^{(m_2)}) \overset{p_2}{\longrightarrow} (\mathbb{T},\sigma^{(m_1+m_2)}).$$

Put

$$\sigma_{m_1+m_2}(\cdot) = \int_{\mathbb{T}\times\mathbb{T}} \overline{\nu}_{m_1,m_2}(\cdot|(\lambda_1,\lambda_2))\,d\sigma^{(m_1)}(\lambda_1)\,d\sigma^{(m_2)}(\lambda_2),$$

where as before $\overline{\nu}_{m_1,m_2}(\cdot|(\lambda_1,\lambda_2)) = \nu_{m_1}(\cdot|\lambda_1)\times\nu_{m_2}(\cdot|\lambda_2)$, and let

$$(\sigma^{(m_1)}\times\sigma^{(m_2)})(\cdot) = \int_{\mathbb{T}}\rho(\cdot|\lambda)\,d\sigma^{(m_1+m_2)}(\lambda).$$

By Lemma 4(i),

$$\nu_{m_1+m_2}(\cdot|\lambda) = \int_{\{(\lambda_1,\lambda_2)\in\mathbb{T}^2:\ \lambda_1\lambda_2=\lambda\}} \nu_{m_1}(\cdot|\lambda_1)\times\nu_{m_2}(\cdot|\lambda_2)\,d\rho((\lambda_1,\lambda_2)|\lambda).$$

In view of Lemma 3, all we need to show is that

$$\widetilde{p_2}(\widetilde{p_{m_1}}(A_{m_1}(\mathcal{P}))\times\widetilde{p_{m_2}}(A_{m_2}(\mathcal{P})))\subset\widetilde{p_{m_1+m_2}}(A_{m_1+m_2}(\mathcal{P}))$$

(here $\widetilde{p_2}$ is defined for $\nu=\sigma^{(m_1)}, \mu=\sigma^{(m_2)}$). Assume that $\lambda\in\widetilde{p_2}(\widetilde{p_{m_1}}(A_{m_1}(\mathcal{P}))\times\widetilde{p_{m_2}}(A_{m_2}(\mathcal{P})))$, that is

$$\rho(\widetilde{p_{m_1}}(A_{m_1}(\mathcal{P}))\times\widetilde{p_{m_2}}(A_{m_2}(\mathcal{P}))|\lambda)>0.$$

We have to show that $\nu_{m_1+m_2}(A_{m_1+m_2}(\mathcal{P})|\lambda)>0$. But by (10) it is sufficient to get that

$$\nu_{m_1+m_2}(A_{m_1}(\mathcal{P})\times A_{m_2}(\mathcal{P})|\lambda)>0.$$

Now,

$$\begin{aligned}
&\nu_{m_1+m_2}(A_{m_1}(\mathcal{P})\times A_{m_2}(\mathcal{P})|\lambda)\\
&=\int_{\{(\lambda_1,\lambda_2)\in\mathbb{T}^2:\ \lambda_1\lambda_2=\lambda\}} \nu_{m_1}(A_{m_1}(\mathcal{P})|\lambda_1)\times\nu_{m_2}(A_{m_2}(\mathcal{P})|\lambda_2)\,d\rho((\lambda_1,\lambda_2)|\lambda)\\
&\geq\int_{\{(\lambda_1,\lambda_2)\in\widetilde{p_{m_1}}(A_{m_1}(\mathcal{P}))\times\widetilde{p_{m_2}}(A_{m_2}(\mathcal{P})):\ \lambda_1\lambda_2=\lambda\}}\\
&\qquad\qquad \nu_{m_1}(A_{m_1}(\mathcal{P})|\lambda_1)\times\nu_{m_2}(A_{m_2}(\mathcal{P})|\lambda_2)d\rho((\lambda_1,\lambda_2)|\lambda)\\
&=\int_{\widetilde{p_{m_1}}(A_{m_1}(\mathcal{P}))\times\widetilde{p_{m_2}}(A_{m_2}(\mathcal{P}))} \nu_{m_1}(A_{m_1}(\mathcal{P})|\lambda_1)\times\nu_{m_2}(A_{m_2}(\mathcal{P})|\lambda_2)\,d\rho((\lambda_1,\lambda_2)|\lambda)\\
&>0.
\end{aligned}$$

The proof of (15) is now complete.

Assume that there exist $1\leq k_1<k_2$ and a nonzero finite measure ν on $\mathbb{T}$ such that

$$\nu\ll\sigma^{(k_i)}|_{\widetilde{p_{k_i}}(A_{k_i}(\mathcal{P}))},\quad i=1,2.$$

Given a natural number $n\geq1$, consider $Q_c^{(nk_1)}$. The maximal spectral type on that space is equal to $\sigma^{(nk_1)}|_{\widetilde{p_{nk_1}}(A_{nk_1}(\mathcal{P}))}$ and moreover, by (15), we have $\nu^{(n)}\ll\sigma^{(nk_1)}|_{\widetilde{p_{nk_1}}(A_{nk_1}(\mathcal{P}))}$. We also have $\nu^{(n-1)}\ll\sigma^{((n-1)k_1)}|_{\widetilde{p_{(n-1)k_1}}(A_{(n-1)k_1}(\mathcal{P}))}$, so by the same argument

$$\nu^{(n)}\ll\sigma^{((n-1)k_1+k_2)}|_{\widetilde{p_{(n-1)k_1+k_2}}(A_{(n-1)k_1+k_2}(\mathcal{P}))};$$

if n is big enough this argument persists and

$$\nu^{(n)} \ll \sigma^{((n-2)k_1+2k_2)}|_{p_{\widetilde{(n-2)k_1}+2k_2}(A_{(n-2)k_1+2k_2}(\mathcal{P}))}.$$

We conclude that the supremum of numbers t_n such that for certain $j_1, \ldots, j_{t_n}$ we have

$$\nu^{(n)} \ll \text{ maximal spectral type of } \mathcal{A}(\mathcal{P}) \text{ on } \mathcal{H}_c^{(j_i)}(\mathcal{P})$$

is equal to infinity and (14) easily follows. ∎

4. On Thouvenot's problem in ergodic theory

We now consider Thouvenot's problem (cf. Introduction). Let B be a Banach space and T an isometry on it. If $S \subset B$ then by $Z(S)$ we mean

$$Z(S) = \operatorname{span}\{fT^i\colon f \in S, i \in \mathbb{Z}\}.$$

The proofs of the following two lemmas are similar to the proofs of the corresponding results from [7] (see also [3]).

LEMMA 6: *Assume that $(B_n)_{n\geq 1}$ is a sequence of closed T-invariant subspaces with*

$$B_n \subset B_{n+1} \quad \textit{and} \quad \overline{\bigcup_{n\geq 1} B_n} = B.$$

Assume moreover that

$$(\exists N \geq 1)(\forall n \geq 1)(\exists f_1^{(n)}, \ldots, f_N^{(n)}) \quad B_n = Z(\{f_1^{(n)}, \ldots, f_N^{(n)}\}). \quad ∎$$

Then $\{(f_1, \ldots, f_N) \in B^N\colon Z(\{f_1, \ldots, f_N\}) = B\}$ is a residual subset of B^N.

LEMMA 7: *Let $T\colon (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism of a standard Borel space. Suppose that there exists a sequence $(H_n)_{n\geq 1} \subset L^2(X, \mu)$ of T-invariant subspaces such that*

(i) $H_n \subset H_{n+1}$, $n \geq 1$,
(ii) $(\exists N \geq 1)(\forall n \geq 1)(\exists f_1^{(n)}, \ldots, f_N^{(n)} \in L^2(X, \mu))$ $H_n = Z(\{f_1^{(n)}, \ldots, f_N^{(n)}\})$ *(in $L^2(X, \mu)$),*
(iii) $(\forall 1 < p < +\infty)$ *the L^2- and L^p-norms are equivalent on $H_n, n \geq 1$,*
(iv) $(\forall 1 < p < +\infty)$ $\bigcup_{n\geq 1} H_n$ *is L^p-dense in $L^p(X, \mu)$.*

Then, for each $1 < p < +\infty$ the L^p-multiplicity of T is at most N. ∎

Example 2: Let $T\colon (X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$ be an ergodic automorphism with discrete spectrum; hence X can be assumed to be a compact metric abelian group and $Tx = x + x_0$ with $\overline{\{nx_0\colon n \in \mathbb{Z}\}} = X$; μ is Haar measure on X. Then

$$\hat{G} = \{\chi_1, \chi_2, \ldots\}$$

is an orthonormal basis of $L^2(X,\mu)$. Put $H_n = \operatorname{span}\{\chi_1,\ldots,\chi_n\}$, $n \geq 1$. Hence H_n is of finite dimension (so all the norms are equivalent) and T restricted to H_n has simple L^2-spectrum. Trigonometric polynomials are dense in $C(X)$, hence in every $L^p(X,\mu), p \geq 1$ and all assumptions of Lemma 7 are satisfied. Therefore, in every L^p $(1 \leq p < +\infty)$ the multiplicity is equal to one.

Let T be a Gaussian automorphism. If we put

$$H_n = \mathcal{H}_c^{(1)} \oplus \cdots \oplus \mathcal{H}_c^{(n)}, \quad n \geq 1,$$

then, as already noticed in [7], (i), (iii), (iv) hold.

PROPOSITION 4: *If $\mathcal{P}$ is a compact subgroup in $C(T|\mathcal{H}_r^{(1)})$ and $\mathcal{A}(\mathcal{P})$ is the corresponding factor, then for all $1 < p < +\infty$ the multiplicities in all $L^p(\mathcal{A}(\mathcal{P}),\mu)$ are the same.*

Proof: We have $L^2(\mathcal{A}(\mathcal{P}),\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_c^{(n)}(\mathcal{P})$, where on each $\mathcal{H}_c^{(n)}(\mathcal{P})$ the L^p-norms are equivalent. Put

$$H_n(\mathcal{P}) = \mathcal{H}_c^{(1)}(\mathcal{P}) \oplus \cdots \oplus \mathcal{H}_c^{(n)}(\mathcal{P}).$$

We obtain that (i) and (iii) of Lemma 7 are satisfied. Suppose that $f \in L^p(\mathcal{A}(\mathcal{P}),\mu)$ and $\varepsilon > 0$. Then there exist $n \geq 1$ and $f_\varepsilon \in H_n$ such that $||f - f_\varepsilon||_{L^p} < \varepsilon$. Moreover

$$||E(f|\mathcal{A}(\mathcal{P})) - E(f_\varepsilon|\mathcal{A}(\mathcal{P}))||_{L^p} \leq ||f - f_\varepsilon||_{L^p},$$

so consequently $||f - E(f_\varepsilon|\mathcal{A}(\mathcal{P}))||_{L^p} < \varepsilon$ and, since $E(f_\varepsilon|\mathcal{A}(\mathcal{P})) \in H_n(\mathcal{P})$ (since $f_\varepsilon \in H_n$, its conditional expectation in L^2 belongs to $H_n(\mathcal{P})$), also (iv) is satisfied.

In view of Theorem 2 (and Lemma 6), on $L^2(\mathcal{A}(\mathcal{P}),\mu)$ either

(16) the multiplicity on $H_n(\mathcal{P})$ grows to infinity when n grows to infinity

or

(17) on each $H_n(\mathcal{P})$ the spectrum is simple.

In the first case, we have a natural projection $(1 < p < 2)$

$$L^p(\mathcal{A}(\mathcal{P}),\mu) \xrightarrow{J_n} H_n(\mathcal{P})$$

(L^p acts on L^q as $(L^q)^*$, hence $L^p(\mathcal{A}(\mathcal{P}))$ can be considered as a family of functionals on $H_n(\mathcal{P})$) which commutes with the corresponding actions of T. Therefore the multiplicity must be infinity.

In case of (17) $(p > 2)$, in view of Lemma 7, we conclude that the L^p-multiplicity is equal to one. ∎

REMARK 3: All the proofs calculating L^p-mutiplicity (see [5], [6], [7] and the examples in this paper) are based on the existence of a sequence of special subspaces. We state as a question whether for each ergodic automorphism T there exists a non-zero T-invariant subspace of $L^2(X,\mu)$ such that all L^p-norms $(1 < p < \infty)$ are equivalent on it.

5. Final remarks

5.1 ISOMORPHISM OF T AND T^{-1}. In [4], the problem of isomorphism between T and its inverse has been considered. This problem is related to the problem of representation of a given T as the composition of a finite number of involutions (see [2]). For example, T and T^{-1} are isomorphic via S satisfying $S^2 =\mathrm{Id}$ iff $T = S_1S_2$, where both S_1 and S_2 are involutions. This property is stable under a passage to certain factors. Namely, if $\mathcal{P} \subset \mathrm{WC}(T)$ is compact then $\mathcal{A} = \mathcal{A}(\mathcal{P})$ is also the composition of two involutions (since $\mathcal{A}(\mathcal{P})$ is simply S-invariant; indeed, $SV = V^{-1}S$ for each $V \in \mathcal{P}$ because $V = \lim_{t\to\infty} T^{n_t}$).

If T: $(X,\mathcal{B},\mu) \longrightarrow (X,\mathcal{B},\mu)$ is a Gaussian automorphism then it is always isomorphic to its inverse via an involution (indeed, take I: $\mathbb{R}^{\mathbb{Z}} \longrightarrow \mathbb{R}^{\mathbb{Z}}$,

$$I(\dots,x_{-2},x_{-1},x_0,x_1,x_2,\dots) = (\dots,x_2,x_1,x_0,x_{-1},x_{-2},\dots),$$

then $IT = T^{-1}I$ and since the correlation matrix is symmetric, I preserves the Gauss measure μ).

PROPOSITION 5: *If $\mathcal{P}$ is a compact subgroup of $C(T|\mathcal{H}_r^{(1)})$ then $\mathcal{A}(\mathcal{P})$ is isomorphic to its inverse by an involution.*

Proof: Let I: $Q_r^{(1)} \longrightarrow Q_r^{(1)}$, $(If)(z) = f(\overline{z})$. Since σ is symmetric, I is unitary and moreover $I^2 =\mathrm{Id}$. Furthermore, $IV = V^{-1}I$. By Proposition 1, I has a unique extension to I: $(X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ with $I^2 = \mathrm{Id}$ and $IT = T^{-1}I$. It remains to show that $\mathcal{A}(\mathcal{P})$ is I-invariant. This is equivalent to saying that $Ig = g^{-1}I$ for each $g \in \mathcal{P}$. We have $|g| = 1$ and $g(\overline{z}) = \overline{g(z)}$ so the result easily follows. ∎

5.2 Embeddability in measurable flows. Let us start with a general remark: if T is weakly mixing, $T = T_1$, where $(T_t)_{t\in\mathbb{R}}$ is a measurable flow, then for no T_t, $t \neq 0$ the closure of $\{T_t^n \colon n \in \mathbb{Z}\}$ is compact.

Let T: $(X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be a Gaussian automorphism and $\mathcal{P}$ a compact subgroup of $C(T|\mathcal{H}_r^{(1)})$. Since

$$\mathbb{R} \ni s \mapsto e^{is(\cdot)} \in L^2(\mathbb{T}, \sigma)$$

is continuous, the corresponding embedding $\mathbb{R} \longrightarrow C(T)$ is also continuous ($e^{is(\cdot)}$ extends to an element of $C(T)$), so T is embeddable in a measurable flow. In order to show that also $\mathcal{A}(\mathcal{P})$ enjoys this property it remains to notice that $e^{is(\cdot)}$ does not act as the identity on $\mathcal{A}(\mathcal{P})$. Because of (1) this is equivalent to saying that $e^{is(\cdot)} \notin \mathcal{P}$. An easy application of the above remark completes the proof.

5.3 A question. It follows from this paper that the compact first chaos factors share the following basic properties of Gaussian automorphisms:

(a) group property of the spectrum, i.e. $\rho * \rho \ll \rho$, where ρ is the maximal spectral type,
(b) the maximal spectral multiplicity is either 1 or infinity,
(c) they are compositions of two involutions,
(d) they are embeddable in measurable flows.

Can there exist a factor of a Gaussian automorphism without one of these properties?

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